Configuration-Dynamics Relationship in Nonlinear Networks

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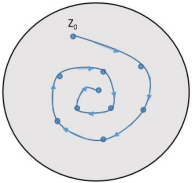
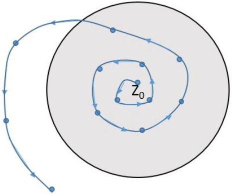
Results for Isolated Nodes

- Given an input, every function produces an output. Consider when the output is re-used as a new input for the *functional node*.
- The behavior of these functional nodes has been studied when these nodes are in isolation (i.e. they only receive input from themselves).
- For $f : \mathbb{C} \to \mathbb{C}$ continuous, the orbit of any $z_0 \in \mathbb{C}$ is the sequence $z_0 \to z_1 = f(z_0) \to z_2 = f^{\circ 2}(z_0) \cdots$





Julia set of a complex map $f : \mathbb{C} \to \mathbb{C}$

z_0 is a **prisoner** of $f \Leftrightarrow \{f^{\circ n}(z_0)\}_{n\in\mathbb{N}}$ is bounded

z_0 is an **escapee** of $f \Leftrightarrow \{f^{\circ n}(z_0)\}_{n\in\mathbb{N}}$ is not bounded

The **prisoner set** of f: $P(f) = \{z_0 \in \mathbb{C} \text{ prisoner }\}$

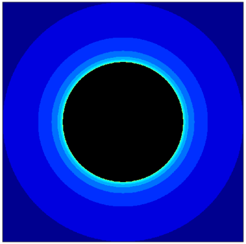
The **escape set** of f: $E(f) = \{z_0 \in \mathbb{C} \text{ escapee }\}$

The **Julia set** of f (Gaston Julia, 1893-1978):

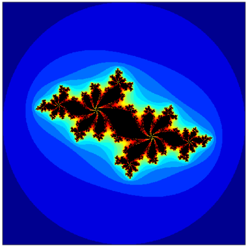
$$J(f) = \partial P(f) = \partial E(f)$$

Julia sets for the logistic family $f_c(z) \to z^2 + c$

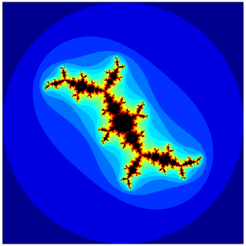
c=0



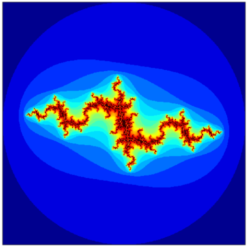
c=-0.62-0.432i



c=-0.117-0.856i

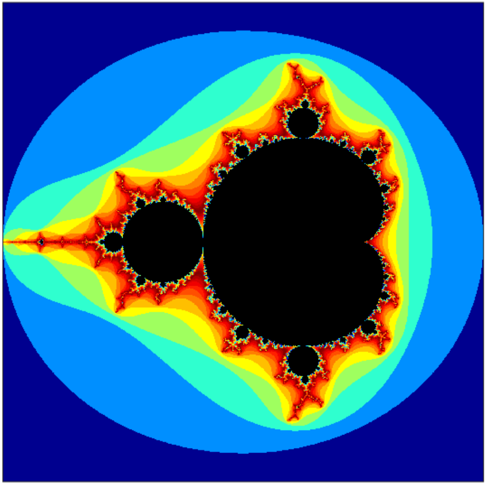


c=-1.18-0.2i



The Mandelbrot set

Definition. $\mathcal{M} = \{c \in \mathbb{C},\ f_c^{\circ n}(0) \text{ bounded }\}$



http://math.bu.edu/DYSYS/applets/JuliaIteration.html

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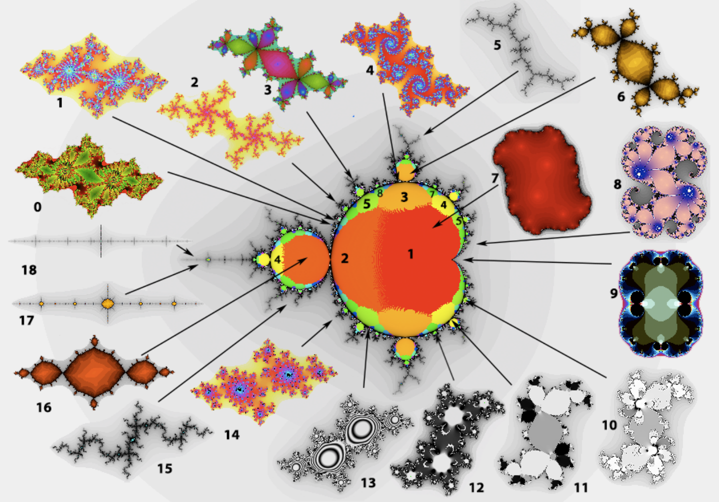

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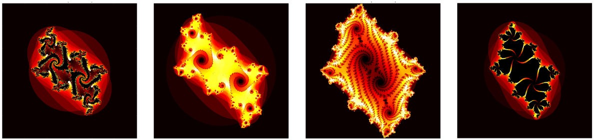
Coupled networks of logistic maps

Many natural systems are organized as self-interacting networks, with each node receiving inputs from both itself and other nodes. We study the temporal behavior of a network in which the nodes are complex logistic maps, coupled according to:

$$z_k(t) \longrightarrow z_k(t+1) \quad = \quad \left(\sum_{j=1}^{n} g_{kj} A_{kj} z_j \right)^2 + c_k, \ \text{for } c_k \in \mathbb{C}$$

A = the graph adjacency matrix; g_{jk} = edge weights

network architecture $\Longrightarrow$ effects on dynamics



Escape Radius for Dominated Networks

Definition. We say that a network is **dominated** with self loops if, for each node $1 \leqslant j \leqslant n$, there exists a node $\sigma(j)$ for which

$$|g_{\sigma(j)j}| > \sum_{l \neq j} |g_{\sigma(j)l}|$$

In other words, each node sends to another node of its choice a projection edge which is stronger than the sum of the strength of all other incoming edges to the receiving node.

Theorem. Dominated networks with identical c values for all nodes have the escape radius property.

Escape Radius for Feed-Forward Networks

Definition. We say that a network is **feed-forward** with self loops if $g_{ii} \neq 0$ for all $1 \leqslant i \leqslant n$, and if for all nodes $1 \leqslant j \leqslant n$ and all iterations $k \geqslant 0$ we have

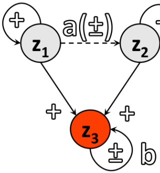
$$z_j(k+1) = \left[\sum_{l \leqslant j} g_{jl} z_l(k)\right]^2 + c$$

(in other words, if its adjacency matrix is lower triangular and has no diagonal zeros).

Theorem. Feed-forward networks with self loops and identical c values for all nodes have the escape radius property.

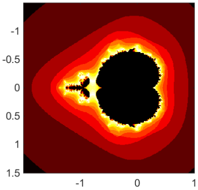
Connectedness of Network Equi-Mandelbrot Sets

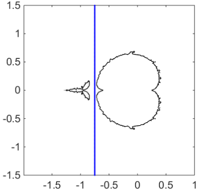
One interesting result from quadratic networks is that many Mandelbrot sets are not connected. E.g., for the network:



$$
\begin{aligned}
z_1 &\rightarrow z_1^2 + c \\
z_2 &\rightarrow (az_1 + z_2)^2 + c \\
z_3 &\rightarrow (z_1 + z_2 + bz_3)^2 + c
\end{aligned}
$$

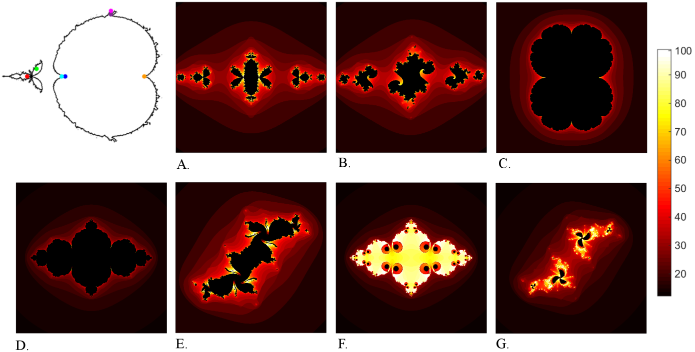
with connectivity weights $a = -2/3$, $b = -1/3$, we have:





Connectedness of Network Uni-Julia Sets

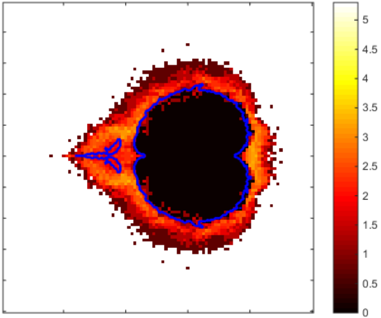
Remark. Many Julia sets are neither connected nor dust.



Uni-Julia sets *for* $a = -2/3$ *and* $b = -1/3$, *for varying* c: **A.** $c = -1$; **B.** $c = -0.9 + 0.08i$; **C.** $c = 0.25$; **D.** $c = -0.595$; **E.** $c = -0.11 + 0.66i$; **F.** $c = -0.63$; **G.** $c = -0.11 + 0.7i$.

Mandelbrot set and connectedness locus

Remark. Many Julia sets are neither connected nor dust. There happens around the boundary of the equi-M set, via Julia sets with various numbers of connected components



Comparison between the equi-M set and the uni-J set connectedness locus. The blue curve corresponds to the boundary of the equi-M sets.

Current and future questions

- **Dimensionality Reduction.** We develop generalized rules and specific cases under which dimensionality reduction (i.e. treating a group of nodes as a single node) is permitted *with preservation of dynamics*.
- **Prediction of Dynamics.** We search for graph features which can be used to predict/classify dynamics for each of the three models.
- **Universality.** We search for graph properties which are both robust within a model and which translate between the different models.

Modelling applications

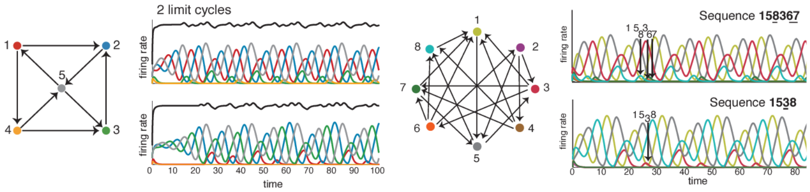
1. **Competitive threshold-linear networks** (TLNs). Models of neural networks consisting of n simple, perceptron-like neurons. (Curto and Morrison, 2018.)
2. **Reduced model of inhibitory clusters** (the RMIC). Model of spiking activity and synchronization in the reticular thalamic nucleus. (Golomb and Rinzel, 1994.)
3. **Chemical oscillatory networks**. Models of photosensitive chemical oscillators which spontaneously form sychronized clusters. (Nkomo, Tinsley, and Showalter, 2013.)

Threshold-Linear Networks

$$\frac{dx_i}{dt} = -x_i + \left[\sum_{j=1}^{n} W_{ij}x_j + b_i\right]$$

- n is the number of neurons
- $x_i(t)$ is the activity level (firing rate) of the i th neuron
- W_{ij} is the connection strength from neuron j to neuron
- $[\cdot]_+ = \mathsf{max}\{\cdot, 0\}$ is the threshold nonlinearity

Threshold-Linear Networks



Although each node has close-to-linear behaviour, TLNs exhibit ensemble nonlinear dynamics determined by connectivity.

Threshold-Linear Networks

Utility in dynamics prediction. Since TLN dynamics are entirely determined by connectivity, we plan to learn more about the relationship between network structure and behavior in hopes of applying it to dynamics prediction in our complex quadratic maps.

Reduced Model of Inhibitory Clusters

Action Potential. In animals, two main types of action potentials:

1. **Na channels** - usually last for very short periods of time (often <1ms)
2. **Ca channels** - can last for 100ms or longer

In some neurons (including neurons in the reticular thalamic nucleus), Ca action potentials (slow spikes) provide a driving force for a long burst of rapid Na spikes

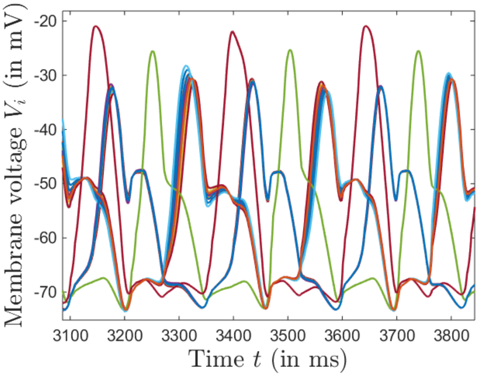
Reduced Model of Inhibitory Clusters

$$
\begin{aligned}
C\frac{dV_i}{dt} &= I_{\text{Ca}}(V_i,h_i) + I_{\text{L}}(V_i) - \frac{g_{\text{syn}}}{n}(V_i - V_{\text{syn}})\sum_{i=1}^{n} s_i(t) \\
\frac{dh_i}{dt} &= k_h(V_i)[h_\infty(V_i) - h_i] \\
\frac{ds_i}{dt} &= k_f \cdot s_\infty(V_i)(1 - s_i) - k_r s_i
\end{aligned}
$$

with the currents given by

$$
\begin{aligned}
I_{\text{Ca}}(V,h) &= -g_{\text{Ca}} m_\infty^3(V) h (V - V_{\text{Ca}}) \text{ and} \\
I_L(V) = &= -g_L(V - V_L)
\end{aligned}
$$

Reduced Model of Inhibitory Clusters



Example of cluster formation for $g_{\text{syn}} = 0.345 \text{ mS/cm}^2$

Reduced Model of Inhibitory Clusters

Utility for Dimensionality Reduction. The spontaneously formed clusters have identical dynamics and function as a single unit, despite being composed of many neurons. The conditions which generate clusters may provide insight into the conditions which permit dimensionality reductions.

Photosensitive Oscillatory Networks

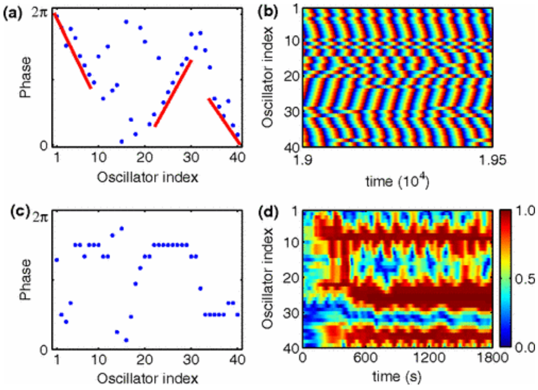
$$\frac{dX_j}{dt} = f(X_j, Z_j, q_j) + \frac{\phi_j}{\epsilon_1}$$
$$\frac{dZ_j}{dt} = g(X_j, Z_j, q_j) + 2\phi_j$$

- f, g are the nonphotochemical reaction components
- X_j = [$HBrO_2$] and Z_j = [$Ru(bpy)^{3+}$]
- q_j is the stoichiometric factor

$$\phi_j = \phi_0 + \sum_{\rho=j-n}^{j+n} K(Z_\rho(t-\tau) - Z_j(t))$$

Photosensitive Oscillatory Networks

Chimera states. These networks spontaneously form groups of coexisting synchronized and unsynchronized oscillators.



Phosensitive Oscillatory Networks

Utility in Dimensionality Reduction. Similar to the RMIC, further observing the conditions under which cluster states form may provide insight in regards to dimensionality reduction.

Chimera states and network dynamics. Additionally, the existence of chimera states may provide a physical analogy for our observation of co-existing prisoner/escapee nodes.

Other models

Note on choice of models. These models were also chosen for their signifigant differences in:

- *the type of node-wise dynamics.* discrete vs continuous, almost-linear vs highly nonlinear, etc
- *the measures used to asses ensemble long-term dynamics.* topology of the asymptotic set, center manifold theory for analysis of equilibria and cycles, synchronization methods

Remark. Discovery of features which translate between these three models are more signifigant with regards to universality because of the signifigant differences between these models.

References and Acknowledgments:

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